GCE

Mathematics (MEI)

Advanced GCE

Unit **4758:** Differential Equations

Mark Scheme for January 2013

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This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by examiners. It does not indicate the details of the discussions which took place at an examiners' meeting before marking commenced.

All examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the report on the examination.

OCR will not enter into any discussion or correspondence in connection with this mark scheme.

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Annotations

Annotation in scoris	Meaning
✓and ✗	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
Highlighting	

Other abbreviations in mark scheme	Meaning
E1	Mark for explaining
U1	Mark for correct units
G1	Mark for a correct feature on a graph
M1 dep*	Method mark dependent on a previous mark, indicated by *
cao	Correct answer only
oe	Or equivalent
rot	Rounded or truncated
soi	Seen or implied
www	Without wrong working

Subject-specific Marking Instructions for GCE Mathematics (MEI) Mechanics strand

a Annotations should be used whenever appropriate during your marking.

The A, M and B annotations must be used on your standardisation scripts for responses that are not awarded either 0 or full marks. It is vital that you annotate standardisation scripts fully to show how the marks have been awarded.

For subsequent marking you must make it clear how you have arrived at the mark you have awarded.

b An element of professional judgement is required in the marking of any written paper. Remember that the mark scheme is designed to assist in marking incorrect solutions. Correct *solutions* leading to correct answers are awarded full marks but work must not be judged on the answer alone, and answers that are given in the question, especially, must be validly obtained; key steps in the working must always be looked at and anything unfamiliar must be investigated thoroughly.

Correct but unfamiliar or unexpected methods are often signalled by a correct result following an *apparently* incorrect method. Such work must be carefully assessed. When a candidate adopts a method which does not correspond to the mark scheme, award marks according to the spirit of the basic scheme; if you are in any doubt whatsoever (especially if several marks or candidates are involved) you should contact your Team Leader.

c The following types of marks are available.

M
A suitable method has been selected and *applied* in a manner which shows that the method is essentially understood. Method marks are not usually lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, eg by substituting the relevant quantities into the formula. In some cases the nature of the errors allowed for the award of an M mark may be specified.

A
Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied). Therefore M0 A1 cannot ever be awarded.

B
Mark for a correct result or statement independent of Method marks.

E
A given result is to be established or a result has to be explained. This usually requires more working or explanation than the establishment of an unknown result.

Unless otherwise indicated, marks once gained cannot subsequently be lost, eg wrong working following a correct form of answer is ignored. Sometimes this is reinforced in the mark scheme by the abbreviation isw. However, this would not apply to a case where a candidate passes through the correct answer as part of a wrong argument.

d When a part of a question has two or more 'method' steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. (The notation 'dep *' is used to indicate that a particular mark is dependent on an earlier, asterisked, mark in the scheme.) Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are successfully run together by the candidate, the earlier marks are implied and full credit must be given.

e The abbreviation ft implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only — differences in notation are of course permitted. A (accuracy) marks are not given for answers obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable. In such cases, exactly what is acceptable will be detailed in the mark scheme rationale. If this is not the case please consult your Team Leader.

Sometimes the answer to one part of a question is used in a later part of the same question. In this case, A marks will often be 'follow through'. In such cases you must ensure that you refer back to the answer of the previous part question even if this is not shown within the image zone. You may find it easier to mark follow through questions candidate-by-candidate rather than question-by-question.

f Unless units are specifically requested, there is no penalty for wrong or missing units as long as the answer is numerically correct and expressed either in SI or in the units of the question. (e.g. lengths will be assumed to be in metres unless in a particular question all the lengths are in km, when this would be assumed to be the unspecified unit.)

We are usually quite flexible about the accuracy to which the final answer is expressed and we do not penalise over-specification.

When a value is given in the paper
Only accept an answer correct to at least as many significant figures as the given value. This rule should be applied to each case.

When a value is not given in the paper
Accept any answer that agrees with the correct value to 2 s.f.

ft should be used so that only one mark is lost for each distinct accuracy error, except for errors due to premature approximation which should be penalised only once in the examination.

There is no penalty for using a wrong value for *g*. E marks will be lost except when results agree to the accuracy required in the question.

g Rules for replaced work

If a candidate attempts a question more than once, and indicates which attempt he/she wishes to be marked, then examiners should do as the candidate requests.

If there are two or more attempts at a question which have not been crossed out, examiners should mark what appears to be the last (complete) attempt and ignore the others.

NB Follow these maths-specific instructions rather than those in the assessor handbook.

h For a *genuine* misreading (of numbers or symbols) which is such that the object and the difficulty of the question remain unaltered, mark according to the scheme but following through from the candidate's data. A penalty is then applied; 1 mark is generally appropriate, though this may differ for some units. This is achieved by withholding one A mark in the question.

Marks designated as cao may be awarded as long as there are no other errors. E marks are lost unless, by chance, the given results are established by equivalent working.

'Fresh starts' will not affect an earlier decision about a misread.

Note that a miscopy of the candidate's own working is not a misread but an accuracy error.

i If a graphical calculator is used, some answers may be obtained with little or no working visible. Allow full marks for correct answers (provided, of course, that there is nothing in the wording of the question specifying that analytical methods are required). Where an answer is wrong but there is some evidence of method, allow appropriate method marks. Wrong answers with no supporting method score zero. If in doubt, consult your Team Leader.

Question			Answer	Marks	Guidance	
1	(i)		$\lambda^3+2\lambda^2-5\lambda-6=0$	M1		
			$2^3+2\times2^2-5\times2-6=0$	E1	Allow if implicit in factorisation of cubic	
			$(\lambda-2)(\lambda+1)(\lambda+3)=0$	M1	Attempt roots (any method)	
			$\lambda=(2),-1,-3$	A1		
			CF $A\mathrm{e}^{2x}+B\mathrm{e}^{-x}+C\mathrm{e}^{-3x}$	F1		
			PI $y=a\sin x+b\cos x$	B1		
			$y'=a\cos x-b\sin x \quad y''=-a\sin x-b\cos x$			
			$y'''=-a\cos x+b\sin x$			
			$(-a\cos x+b\sin x)+2(-a\sin x-b\cos x)$	M1	Differentiate and substitute	
			$-5(a\cos x-b\sin x)-6(a\sin x+b\cos x)=\sin x$			
			$\left.\begin{array}{l}-6a-8b=0\\-8a+6b=1\end{array}\right\}\Rightarrow a=-\frac{2}{25},b=\frac{3}{50}$	M1 A1	Compare coefficients and solve	
			GS $y=\frac{3}{50}\cos x-\frac{2}{25}\sin x+A\mathrm{e}^{2x}+B\mathrm{e}^{-x}+C\mathrm{e}^{-3x}$	F1		
				[10]		
1	(ii)		bounded so $A=0$	F1		
			$x=0,y=1\Rightarrow 1=\frac{3}{50}+B+C$	F1	Use condition	
			$y'=-\frac{3}{50}\sin x-\frac{2}{25}\cos x-B\mathrm{e}^{-x}-3C\mathrm{e}^{-3x}$	M1	Differentiate	
			$x=0,y'=0\Rightarrow 0=-\frac{2}{25}-B-3C$	M1	Use condition	
			$B=\frac{29}{20},C=-\frac{51}{100}$	A1		
			$y=\frac{3}{50}\cos x-\frac{2}{25}\sin x+\frac{29}{20}\mathrm{e}^{-x}-\frac{51}{100}\mathrm{e}^{-3x}$	F1		
				[6]		
1	(iii)		$y\approx\frac{3}{50}\cos x-\frac{2}{25}\sin x$	F1		
			amplitude $=\frac{1}{50}\sqrt{3^2+4^2}=\frac{1}{10}$	M1 A1		
				B1	Sketch showing oscillations with their amplitude. More than one oscillation, ignore origin	
				[4]		

Question			Answer	Marks	Guidance	
1	(iv)		bounded so $B=C=0$	B1		
			$x=0, y=1 \Rightarrow 1=\frac{3}{50}+A \Rightarrow A=\frac{47}{50}$	M1		
			$x=0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2}{25}+2\left(\frac{47}{50}\right)\neq 0$	M1	Or $A=\frac{1}{25}$	
			So no such solution	A1 **[4]**	www	
2	(i)		N2L: $m\frac{\mathrm{d}v}{\mathrm{d}t}=9.8m-mkv \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t}=9.8-kv$	E1		
			EITHER $\int\frac{1}{9.8-kv}\,\mathrm{d}v=\int\mathrm{d}t$	M1	Separate and integrate	
			$-\frac{1}{k}\ln\lvert 9.8-kv\rvert=t+c$	A1 A1	LHS RHS (including constant on one side)	
			$9.8-kv=A\mathrm{e}^{-kt}$	M1	Rearrange, dealing properly with constant	
			$t=0, v=0 \Rightarrow 9.8=A$	M1	Use condition	
			$v=\frac{9.8}{k}\left(1-\mathrm{e}^{-kt}\right)$	A1	cao	
			OR Integrating factor e^{kt}	M1		
			$v\mathrm{e}^{kt}=\int 9.8\mathrm{e}^{kt}\,\mathrm{d}t$	A1	Multiply both sides by IF and recognise derivative on LHS	
			$v\mathrm{e}^{kt}=\frac{9.8}{k}\mathrm{e}^{kt}+A$	M1	Integrate both sides	
				A1	Must include constant	
			$t=0, v=0 \Rightarrow A=-\frac{9.8}{k}$	M1	Use condition	
			$v=\frac{9.8}{k}\left(1-\mathrm{e}^{-kt}\right)$	A1	cao	
			OR Auxiliary equation $\lambda+k=0$	M1		
			CF $v=A\mathrm{e}^{-kt}$	A1		
			PI $v=b, v'=0 \therefore b=\frac{9.8}{k}$	M1		

Question			Answer	Marks	Guidance	
			GS $v = Ae^{-kt} + \frac{9.8}{k}$	A1		
			$t = 0, v = 0 \Rightarrow A = -\frac{9.8}{k}$	M1	Use condition	
			$v = \frac{9.8}{k}\left(1 - e^{-kt}\right)$	A1	cao	
				[7]		
2	**(ii)**		$k = \frac{9.8}{7} = 1.4$	B1		
				[1]		
2	**(iii)**		$m\frac{dv}{dt} = 9.8m - 0.2mv^2$	M1		
			$\int \frac{1}{9.8 - 0.2v^2}\, dv = \int dt$	M1		
			$\int \frac{5}{49 - v^2}\, dv = t + c_2$	A1	RHS (including constant on one side)	
			$\frac{5}{14}\int\left(\frac{1}{7+v} + \frac{1}{7-v}\right)dv = t + c_2$	M1	Integrate	
			$\frac{5}{14}\left(\ln\|7+v\| - \ln\|7-v\|\right) = t + c_2$	A1	LHS	
			$\ln\left\|\frac{7+v}{7-v}\right\| = \frac{14}{5}(t + c_2)$			
			$\frac{7+v}{7-v} = Be^{14t/5}$	M1	Rearrange into a form without ln, dealing properly with constant	
			$t = 0, v = 0 \Rightarrow B = 1$	M1	Use condition	
			$7 + v = e^{14t/5}(7 - v)$	M1	Rearrange to get v in terms of t	
			$v = 7\left(\frac{e^{14t/5} - 1}{e^{14t/5} + 1}\right) = 7\left(\frac{1 - e^{-14t/5}}{1 + e^{-14t/5}}\right)$	A1	oe	
			as $t \to \infty, v \to 7\left(\frac{1-0}{1+0}\right) = 7$	E1		
				[10]		

Question			Answer	Marks	Guidance	
2	(iv)		$\dot{v} = g - 0.529v^{3/2}$	E1		
			t | v | $\dot{v}$ | $h\dot{v}$	M1	Use algorithm	
			0 | 0 | 9.8 | 0.98	A1	$v(0.1)$	
			0.1 | 0.98 | 9.2868 | 0.92868	A1	$\dot{v}(0.1)$ at least 3d.p.	
			0.2 | 1.9087 | |	A1	$v(0.2) = 1.91$ to 3s.f.	
				[5]		
2	(v)		$\dot{v} = 0 \Rightarrow g = 0.529v^{3/2} \Rightarrow v = 7.00$ (3 sf)	E1	Or $9.8 - 0.529 \times 7^{3/2} \approx 0$	
				[1]		
3	(a)		$I = \exp\left(\int -\tan x \, dx\right)$	M1		
			$= \exp\left(-\ln\lvert\sec x\rvert\right)$ or $\exp\left(\ln\lvert\cos x\rvert\right)$	A1		
			$= \cos x$	A1		
			$\cos x \frac{dy}{dx} - y\sin x = \sin x \cos x$			
			$\frac{d}{dx}\left(y\cos x\right) = \sin x \cos x$	M1	Multiply and recognise derivative	
			$y\cos x = \int \sin x \cos x \, dx$	M1	Attempt integral	
			$= \int \frac{1}{2}\sin 2x \, dx$	M1	Use identity, substitution or inspection on RHS	
			$= -\frac{1}{4}\cos 2x + c_1$ (or $\frac{1}{2}\sin^2 x + k$)	A1	oe (but must include constant)	
			$x = 0, y = 1 \Rightarrow 1 = -\frac{1}{4} + c_1$	M1	Use condition	
			$y = \frac{5 - \cos 2x}{4\cos x}$ or $y = \frac{\sin^2 x + 2}{2\cos x}$ or $y = \frac{3 - \cos^2 x}{2\cos x}$	A1	oe	
				[9]		

Question			Answer	Marks	Guidance	
3	(b)		$p'(x)+f(x)p(x)=g(x)$	M1	Must be $p'(x)$ oe	
			$c'(x)+f(x)c(x)=0$	M1	Must be $c'(x)$ oe	
			$\frac{dy}{dx}+f(x)y=p'(x)+Ac'(x)+f(x)\left(p(x)+Ac(x)\right)$	M1	Substitute in DE	
			$=p'(x)+f(x)p(x)+A\left(c'(x)+f(x)c(x)\right)$	M1	Separate p and c terms	
			$=g(x)+A\times 0=g(x)$	E1	Complete argument	
				[5]		
3	(c)	(i)	$y=e^{x^2}\Rightarrow\frac{dy}{dx}=2xe^{x^2}$	B1		
			so LHS of DE $=2xe^{x^2}+\frac{2}{x}e^{x^2}$	M1		
			$=2e^{x^2}\left(x+\frac{1}{x}\right)=2e^{x^2}\left(\frac{x^2+1}{x}\right)$	E1		
				[3]		
3	(c)	(ii)	$\frac{dy}{dx}=-\frac{2y}{x}\Rightarrow\int\frac{1}{y}dy=\int-\frac{2}{x}dx$	M1		
			$\ln\lvert y\rvert=-2\ln\lvert x\rvert+c_2$	A1	LHS	
				A1	RHS including constant	
			$y=Ax^{-2}$	A1	cao	
			OR Integrating factor = x^2	B1		
			$\frac{d}{dx}\left(yx^2\right)=0$	M1		
			$yx^2=A$	A1		
			$y=Ax^{-2}$	A1	cao	
				[4]		

Question			Answer	Marks	Guidance	
3	(c)	(iii)	$y = e^{x^2} + Ax^{-2}$	B1	Here A combines the arbitrary constants of **(b)** and **(c) (ii)** into a single arbitrary constant.	
			$1 = e^1 + A$	M1	Use condition	
			$y = e^{x^2} + (1-e)x^{-2}$	A1		
				[3]		
4	(i)		$y = \frac{2}{3}\left(-\dot{x} - \frac{1}{2}x + t\right)$	M1		
			$\dot{y} = \frac{2}{3}\left(-\ddot{x} - \frac{1}{2}\dot{x} + 1\right)$	M1	Differentiate	
			$\frac{2}{3}\left(-\ddot{x} - \frac{1}{2}\dot{x} + 1\right) = \frac{3}{2}x - \frac{1}{2}\cdot\frac{2}{3}\left(-\dot{x} - \frac{1}{2}x + t\right) + 2t$	M1	Substitute	
			$2\ddot{x} + 2\dot{x} + 5x = 2 - 5t$	A1	oe	
			AE $2\lambda^2 + 2\lambda + 5 = 0$	M1		
			$\lambda = -\frac{1}{2} \pm \frac{3}{2}\mathrm{j}$	A1		
			CF $e^{-\frac{1}{2}t}\left(A\cos\frac{3}{2}t + B\sin\frac{3}{2}t\right)$	M1	Correct form	
				F1	FT wrong roots	
			PI $x = a + bt$	B1		
			$\dot{x} = b,\ \ddot{x} = 0 \Rightarrow 2b + 5(a + bt) = 2 - 5t$	M1	Differentiate and substitute	
			$\left.\begin{array}{r} 2b + 5a = 2 \\ 5b = -5 \end{array}\right\} \Rightarrow a = \frac{4}{5}, b = -1$	M1 A1	Equate coefficients and solve	
			GS $x = \frac{4}{5} - t + e^{-\frac{1}{2}t}\left(A\cos\frac{3}{2}t + B\sin\frac{3}{2}t\right)$	F1		
				[13]		

Question		Answer	Marks	Guidance
4	(ii)	$y=\frac{2}{3}\left(-\dot{x}-\frac{1}{2}x+t\right)$	M1	
		$\dot{x}=-1-\frac{1}{2}\mathrm{e}^{-\frac{1}{2}t}\left(A\cos\frac{3}{2}t+B\sin\frac{3}{2}t\right)$	M1	Must be using product rule
		$+\mathrm{e}^{-\frac{1}{2}t}\left(-\frac{3}{2}A\sin\frac{3}{2}t+\frac{3}{2}B\cos\frac{3}{2}t\right)$	F1	Must be GS from **(i)**
		$y=\frac{2}{5}+t+\mathrm{e}^{-\frac{1}{2}t}\left(A\sin\frac{3}{2}t-B\cos\frac{3}{2}t\right)$	A1	
			[4]	
4	(iii)	$x=1,t=0\Rightarrow 1=\frac{4}{5}+A\Rightarrow A=\frac{1}{5}$	M1	
		$y=0,t=0\Rightarrow 0=\frac{2}{5}-B\Rightarrow B=\frac{2}{5}$	M1	
		$x=\frac{4}{5}-t+\mathrm{e}^{-\frac{1}{2}t}\left(\frac{1}{5}\cos\frac{3}{2}t+\frac{2}{5}\sin\frac{3}{2}t\right)$		
		$y=\frac{2}{5}+t+\mathrm{e}^{-\frac{1}{2}t}\left(\frac{1}{5}\sin\frac{3}{2}t-\frac{2}{5}\cos\frac{3}{2}t\right)$	A1	Both
			[3]	
4	(iv)	$x+y=\frac{6}{5}+\mathrm{e}^{-\frac{1}{2}t}\left(\frac{3}{5}\sin\frac{3}{2}t-\frac{1}{5}\cos\frac{3}{2}t\right)$	M1	Adding and attempting the limit
		$t\to\infty\Rightarrow \mathrm{e}^{-\frac{1}{2}t}\to 0\Rightarrow x+y\to\frac{6}{5}$	E1	FT for finite limit
		$x+y=\frac{6}{5}\Leftrightarrow\frac{3}{5}\sin\frac{3}{2}t-\frac{1}{5}\cos\frac{3}{2}t=0\Leftrightarrow\tan\frac{3}{2}t=\frac{1}{3}$	M1	Establish equation and indicate method
		which occurs (infinitely often)	E1	Correctly investigate the existence of a solution, but explicit solution for t not required.
			[4]	

OCR (Oxford Cambridge and RSA Examinations)
1 Hills Road
Cambridge
CB1 2EU

OCR Customer Contact Centre

Education and Learning
Telephone: 01223 553998
Facsimile: 01223 552627
Email: general.qualifications@ocr.org.uk

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Head office
Telephone: 01223 552552
Facsimile: 01223 552553

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