GCE

Mathematics (MEI)

Advanced GCE **4758**

Differential Equations

Mark Scheme for June 2010

Oxford Cambridge and RSA Examinations

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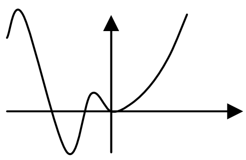
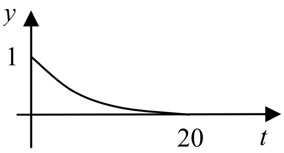
1(i)	$\alpha^2+4\alpha+8=0$	M1	Auxiliary equation	
	$\alpha=-2\pm 2\mathrm{j}$	A1		
	CF $\mathrm{e}^{-2x}(A\cos 2x+B\sin 2x)$	M1	CF for complex roots	
		F1	CF for their roots	
	PI $y=ax^2+bx+c$	B1		
	$\dot{y}=2ax+b, \ddot{y}=2a$			
	$2a+4(2ax+b)+8(ax^2+bx+c)=32x^2$	M1	Differentiate twice and substitute	
	$8a=32$	M1	Compare coefficients	
	$8a+8b=0$			
	$2a+4b+8c=0$	M1	Solve	
	$a=4, b=-4, c=1$	A1		
	GS $y=4x^2-4x+1+\mathrm{e}^{-2x}(A\cos 2x+B\sin 2x)$	F1	PI + CF with two arbitrary constants	10
(ii)	$x=0, y=0 \Rightarrow A=-1$	M1	Use condition	
	$y'=8x-4+\mathrm{e}^{-2x}(-2A\sin 2x+2B\cos 2x$ $-2A\cos 2x-2B\sin 2x)$	M1	Differentiate (product rule)	
	$x=0, y'=0 \Rightarrow 0=-4+(2B-2A) \Rightarrow B=1$	M1	Use condition	
	$y=4x^2-4x+1+\mathrm{e}^{-2x}(\sin 2x-\cos 2x)$	A1	Cao	4
(iii)	$x\to-\infty \Rightarrow y$ oscillates	B1	Oscillates	
	With (exponentially) growing amplitude	B1	Amplitude growing	2
(iv)	$y\sim(2x-1)^2$ or $4x^2-4x+1$	B1		
				1
(v)		B1	Minimum point at origin	
		B1	Oscillates for $x<0$ with growing amplitude	
		B1	Approximately parabolic for $x>0$	
				3
(vi)	At stationary point $\frac{\mathrm{d}y}{\mathrm{d}x}=0$	M1	Set first derivative (only) to zero in DE	
	So $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=32x^2-8y$	A1		
	$y<0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}>0$	M1	Deduce sign of second derivative	
	$\Rightarrow$ minimum	E1	Complete argument	4

2(a)(i)	$IF = \exp\int 2\mathrm{d}t$	M1	Attempt IF	
	$= \mathrm{e}^{2t}$	A1		
	$\mathrm{e}^{2t}\dfrac{\mathrm{d}y}{\mathrm{d}t} + 2\mathrm{e}^{2t}y = 1$	M1*	Multiply by IF	
	$\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2t}y) = 1$	A1		
	$\mathrm{e}^{2t}y = t + A$	*M1A1	Integrate both sides	6
	$[y = \mathrm{e}^{-2t}(t + A)]$			
	Alternative method: CF $y = E\mathrm{e}^{-2t}$ PI $y = Ft\mathrm{e}^{-2t}$ In DE: $\mathrm{e}^{-2t}(F - 2Ft) + 2Ft\mathrm{e}^{-2t} = \mathrm{e}^{-2t}$ $F = 1$ $y = \mathrm{e}^{-2t}(t + E)$	B1 B1 M1 M1A1 F1		
(ii)	$\dfrac{\mathrm{d}z}{\mathrm{d}t} + 2z = \mathrm{e}^{-2t}(t + A)$			
	$I = \mathrm{e}^{2t}$	B1	Correct or follows (i)	
	$\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{2t}z) = t + A$	M1	Multiply by IF and integrate	
	$\mathrm{e}^{2t}z = \frac{1}{2}t^2 + At + B$	A1		
	$z = \mathrm{e}^{-2t}(\frac{1}{2}t^2 + At + B)$			
	$t = 0, z = 1 \Rightarrow 1 = B$	M1	Use condition	
	$\dot{z} = -2\mathrm{e}^{-t}(\frac{1}{2}t^2 + At + B) + \mathrm{e}^{-2t}(t + A)$	M1	Differentiate (product rule)	
	$t = 0, \dot{z} = 0 \Rightarrow 0 = -2B + A \Rightarrow A = 2$	M1	Use condition	
	$z = \mathrm{e}^{-2t}(\frac{1}{2}t^2 + 2t + 1)$	A1		7
	Alternative method: PI $x = (Pt + Qt^2)\mathrm{e}^{-2t}$ $P = A$ and $Q = 0.5$ $z = \mathrm{e}^{-2t}(\frac{1}{2}t^2 + At + B)$ Then as above	B1 M1A1	Correct form of PI Complete method	
(b)(i)	$\alpha + 2 = 0 \Rightarrow \alpha = -2$			
	CF $x = C\mathrm{e}^{-2t}$	B1	CF correct	
	PI $x = a\sin t + b\cos t$	B1	Correct form of PI	
	$\dot{x} = a\cos t - b\sin t$			
	In DE: $a\cos t - b\sin t + 2a\sin t + 2b\cos t = \sin t$	M1	Differentiate and substitute	
	$a + 2b = 0, -b + 2a = 1$	M1	Compare and solve	
	$\Rightarrow a = \frac{2}{5}, b = -\frac{1}{5}$	A1		
	GS $x = \frac{1}{5}(2\sin t - \cos t) + C\mathrm{e}^{-2t}$	F1	Their PI + CF	6
(ii)	$\dot{x} = 0, t = 0 \Rightarrow x = 0$ (from DE)	M1	Or differentiate	
	$0 = -\frac{1}{5} + C$	M1	Use condition	
	$x = \frac{1}{5}(2\sin t - \cos t + \mathrm{e}^{-2t}$	A1		3
(iii)	For large t, $x \approx \frac{1}{5}(2\sin t - \cos t) = \frac{1}{5}\sqrt{5}\sin(t\text{-}\phi)$	M1	Complete method	
	So x varies between $-\frac{1}{5}\sqrt{5}$ and $\frac{1}{5}\sqrt{5}$	A1	Accept $\lvert x\rvert \le \frac{1}{5}\sqrt{5}$	2

3(i)	$\int y^{-\frac{1}{2}}\mathrm{d}y = \int -k\mathrm{d}t$	M1	Separate and integrate	
	$2y^{\frac{1}{2}} = -kt + B$	A1	LHS	
		A1	RHS	
	$t = 0, y = 1 \Rightarrow 2 = B$	M1	Use condition	
	$t = 2, y = 0.81 \Rightarrow 1.8 = -2k + 2$	M1	Use condition	
	$\Rightarrow k = 0.1$	A1		
	$y^{\frac{1}{2}} = 1 = 0.05t$			
	$y = (1 - 0.05t)^2$	A1		
	Valid for $1 - 0.05t \geq 0$, i.e. $t \leq 20$	B1√	√ on arithmetical error in k	
		B1 B1	Shape Intercepts	
				10
(ii)	$\int \pi y^{\frac{3}{2}}\mathrm{d}y = \int -0.4\mathrm{d}t$	M1	Separate and integrate	
	$\frac{2}{5}\pi y^{\frac{5}{2}} = -0.4t + C$	A1	LHS	
		A1	RHS	
	$t = 0, y = 1 \Rightarrow C = \frac{2}{5}\pi$	M1	Use condition	
	$y = 0.81 \Rightarrow t = 1.287$	A1		5
(iii)	$\dot{y} = -\dfrac{0.4\sqrt{y}}{\pi(2y - y^2)}$	M1	Rearrange (implied by correct values)	

t	y	$\dot{y}$	$h\dot{y}$			
				M1	Use algorithm	
0	1	–0.12732	–0.01273	A1	$y(0.1)$ (awrt 0.987)	
0.1	0.987268	–0.12653	–0.01265	M1	Use algorithm	
0.2	0.974614			A1	$y(0.2)$ (0.974 to 0.975)	5

(iv)	If $V =$ volume, $v =$ velocity, $A =$ horizontal cross-sectional area,			
	then $\dfrac{\mathrm{d}V}{\mathrm{d}t} = -k_1 v$	M1	Rate of change of volume	
	$v = k_2\sqrt{y}$			
	$A\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{dV}}{\mathrm{d}t}$	M1	Relate rates of change of y and volume	
	$\Rightarrow A\dfrac{\mathrm{d}y}{\mathrm{d}t} = -k_1 k_2\sqrt{y}$	M1	Eliminate volume and/or velocity	
	$\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}t} = -k\sqrt{y}$	E1	Complete argument	4

4(i)	$5y = 2x + 9e^{-2t} - \dot{x}$	M1	y or $5y$ in terms of $x, \dot{x}$	
	$5\dot{y} = 2\dot{x} - 18e^{-2t} - \ddot{x}$	M1	Differentiate	
	$\frac{1}{5}(2\dot{x} - 18e^{-2t} - \ddot{x})$	M1	Substitute for y	
	$= x - \frac{4}{5}(2x + 9e^{-2t} - \dot{x}) + 3e^{-2t}$	M1	Substitute for $\dot{y}$	
	$\Rightarrow \ddot{x} + 2\dot{x} - 3x = 3e^{-2t}$	E1		5
(ii)	$\alpha^2 + 2\alpha - 3 = 0$	M1	Auxiliary equation	
	$\Rightarrow \alpha = 1, -3$	A1		
	CF $Ae^{t} + Be^{-3t}$	F1	CF for their roots	
	PI $x = ae^{-2t}$	B1	PI of correct form	
	$\dot{x} = -2ae^{-2t}, \ddot{x} = 4ae^{-2t}$	M1	Differentiate and substitute	
	$(4a - 4a - 3a)e^{-2t} = 3e^{-2t}$	M1	Compare coefficients and solve	
	$a = -1$	A1		
	GS $x = Ae^{t} + Be^{-3t} - e^{-2t}$	F1	PI + CF with two arbitrary constants	8
(iii)	$y = \frac{1}{5}(2x + 9e^{-2t} - \dot{x})$	M1		
	$\frac{1}{5}(2Ae^{t} + 2Be^{-3t} - 2e^{-2t} + 9e^{-2t}$	M1	Differentiate and substitute	
	$-(Ae^{t} - 3Be^{-3t} + 2e^{-2t}))$	F1	Expression for $\dot{x}$ follows their GS	
	$y = \frac{1}{5}Ae^{t} + Be^{-3t} + e^{-2t}$	A1		4
(iv)	$t = 0, x = 0 \Rightarrow 0 = A + B - 1$	M1	Use condition	
	$t = 0, y = 2 \Rightarrow 2 = \frac{1}{5}A + B + 1$	M1	Use condition	
	$\Rightarrow A = 0, B = 1$			
	$x = e^{-3t} - e^{-2t}$	A1		
	$y = e^{-3t} + e^{-2t}$	A1		4
(v)	As $t \to \infty, x \to 0, y \to 0$	B1		
	$y(0) < 2 \Rightarrow A > 0$	M1	Consider coefficient(s) of e^{t} and mention of $y < 2$	
	$x, y \to \infty$ as $t \to \infty$	E1	Complete argument	3

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