# OCR - Oxford Cambridge and RSATeacher Delivery Guide Pure Mathematics: Proof

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| **Specification** | **Ref.** | **Learning outcomes** | **Notes** | **Notation** | **Exclusions** |
| **PURE MATHEMATICS: PROOF (1)** |
| Proof | Mp1 | Understand and be able to use the structure of mathematical proof.Use methods of proof, including proof by deduction and proof by exhaustion. | Proceeding from given assumptions through a series of logical steps to a conclusion. |  |  |
| p2 | Be able to disprove a conjecture by the use of a counter example. |  |  |  |
| **PURE MATHEMATICS: PROOF (2)** |
| Proof  | p3 | Understand and be able to use proof by contradiction. | Including proof of the irrationality of √2 and the infinity of primes, and application to unfamiliar proofs. |  |  |

# Thinking Conceptually

The key concepts in proof by deduction, and all direct proof, are that we begin with an assumption and use a chain of reasoning to show a result. Note that a detailed approach is required; this will often be algebraic. For proof by exhaustion the key concepts are demonstrating a result by verifying every possible example. Once every possible example is shown to be true then the proof is complete. As guidance for approaches to teaching direct proof we recommend the following stages: make conjectures and generalisations; check the conjectures/generalisations; where appropriate, express statements algebraically and explore strategies for proof; familiarity with famous proofs; focus on algebraic techniques; study commonly used strategies; arrange the steps of a given proof into the right order; provide examples and then both practice and discussion.

For mathematical logic the key concepts involved are *if … then* statements; *if and only if* (iff) statements and, finally, converse statements. The recommended approach to teaching this content is to study the formal ideas within logic, such as implication and the idea of iff; to make note that implication cannot usually be reversed and for students to explore both *if … then* and iff examples and proofs.

The key concept when attempting a disproof is the notion of a *counter example*. The *counter example* is one given instance or number which does not fulfil the original statement. This therefore disproves the theorem or result. As an introduction to teaching it can be useful initially to give non-mathematical examples such as a *left-handed man* to disprove ‘*all men are right-handed’.*

In proof by contradiction the key idea is to assume the opposite of the result which needs to be proven, find a contradiction, showing that the initial assumption is incorrect and therefore proving the result. Approaches to teaching should include the study of famous proofs by contradiction, including the *irrationality of * and the *infinity of the primes.* Both of these proofs can be found within the p3 activity listings.

Common difficulties which learners may have in direct proof include how to adopt a formal approach; avoidance of error; lack of attention to detail; inability to complete; inability to identify how a result may be proven; use of an incorrect argument; lack of fluency with the different types of proof; encountering extremely hard proofs and developing strategies, which may be time consuming.

Common misconceptions with direct proof may include the notion that numerical examples are sufficient, whereas in general an algebraic approach is necessary. For disproof/proof by exhaustion a numerical approach is often sufficient. Students often underestimate the necessary detail required and there is a common belief that a proof should be solved easily and quickly. However it is normal to fail at proof sometimes and confidence generally comes with experience.

Common misconceptions or difficulties in mathematical logic usually lie with iff statements whose proof is stricter. When disproving a result students may be unfamiliar with the notion of a counter example. If proving by contradiction students may not find the approach intuitively obvious. They will need to see at least three examples before they attempt their own proofs.

Conceptual links to other areas of the specification – useful ways to approach this topic to set learners up for topics later in the course:

The concepts in direct proof can be found in the following content areas: differentiation from first principles; trigonometric identities; logarithms; properties of numbers. When teaching proof in these areas the following approaches can be useful: demonstrating the inadequacy of a numerical approach; introducing simple algebraic forms such as even is , odd is , a multiple of three is ; revising GCSE proof; working with sequences in a spreadsheet or with sliders in GeoGebra leading to conjectures and generalisations. Examples of some accessible proofs are: proving the quadratic formula by completing the square; proving the result for the line of symmetry of a quadratic via or by completing the square; practising some of Euclid’s proofs.

Concepts in mathematical logic are encountered in, for example, set theory and identities. Also *implication* in mathematics is used in many forms of mathematical reasoning. Useful approaches here include practising formal methods; practising iff using both the implication and reverse argument; general discussion regarding implication and iff.

When dealing with disproving a result the concepts can be found for example in trigonometry, equation solving, uses of the discriminant and number sets. The *counter example* is frequently used to clear up misconceptions about matrix and vector properties; geometric truths; factors and integration as the reverse of differentiation. It may be helpful to discuss *counter examples* to clarify concepts before beginning disproof. For example, if students think that a polygon with equal sides must be regular, introduce them to a polygon with equal sides which is not regular – this will correct/clarify their conception of what a regular polygon is as well as introducing them to the idea of a counterexample.

Proof by contradiction is linked with concepts found in areas such as number; matrix results and geometry. Useful approaches to reinforce the concepts are the study of famous proofs in class; allowing students to explore proofs without time constraint; group work.

# Thinking Contextually

Proof by deduction can be practised in contexts such as: properties of graphs; trigonometric identities; logarithms; differentiation from first principles; vector results; probability results and series formulae.

Generally in mathematics proof by exhaustion would be used to establish results such as the number property *there are exactly eight primes less than 20* and also some probabilities or binomial coefficients.

Mathematical logic is used for example to practise proofs of *if … then* using geometry, trigonometry or vectors, and to prove *if and only if* (iff) statements linked with theorems concerning primes, even numbers or geometry.

Good contexts for practising disproof are, for example, number theory – primes, evens, odds and irrationals. Also vectors, where using a counter example can show non-parallel; non-orthogonal; non-unit vector. Using a counter example can also be used to demonstrate incorrect integration or non-divisibility.

Finally, for proof by contradiction, good example areas could be number results such as proofs about prime numbers. Some geometric results may be proved by this method.

Many results in Statistics and Mechanics are useful for practising proof, particularly the latter. Simply asking students to show given results, or to justify their working, is enough to develop many of the ideas and techniques.

Proof is developed in Further Mathematics, both within the mandatory pure content and in aspects of the optional content.

# Resources

| **Title** | **Organisation** | **Description** | **Ref** |
| --- | --- | --- | --- |
| [The Greek Legacy: How the Ancient Greeks shaped modern mathematics](https://www.youtube.com/watch?v=y1lIdkoIn0Y)  | Royal Institution | 2 minute video which may help to give a broad overview of some of the ideas within proof. May be a good starter resource. | p1 |
| [Introduction to proof.](https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2010/video-lectures/lecture-1-introduction-and-proofs/) | MIT | This resource is an MIT lecture which introduces some of the ideas within proof.The lecturer is Tom Leighton and the video lasts for 45 minutes. Learners will encounter proving a proposition and, in particular, disproving a proposition by finding a counterexample.  | p1, p2 and p3 |
| [Godel's Incompleteness Theorems](http://www.bbc.co.uk/programmes/b00dshx3) | BBC | Accessible and interesting discussion of mathematics which focuses very much on proof and logic. 42 minutes duration. | p1 |
| [Types of proof and disproof](https://www.tes.com/teaching-resource/types-of-proof-and-disproof-6146814) | TES | 5 worksheets on different types of proof. | p1, p2 and p3 |
| [Alternate Formula for Standard Deviation](http://dsearls.org/courses/M120Concepts/ClassNotes/Statistics/510B2_derivation.htm) | Asbury University | Example proof in context. Could be used for flipped learning. | p1 |
| [Alternate Segment Theorem (proof)](https://www.geogebra.org/m/wK5z9XwD)  | Geogebra | Example of a geometric proof. Accessible and suitable to use at the beginning of your work on proof. Could be a starter activity or used for flipped learning. | p1 |
| [Tetra Perp](http://nrich.maths.org/1963) | nrich | Geometric proof activity which can be solved using vectors. Worked solution provided. Ideal for flipped learning. | p1 |
| [Eulid’s Elements Book 1](http://aleph0.clarku.edu/~djoyce/elements/bookI/propI1.html) | Clark University | Online version of Euclid’s Elements in full with proofs. Suitable resource for flipped learning. | p1 |
| [Can we prove these four binomial coefficient identities?](https://undergroundmathematics.org/counting-and-binomials/r6102) | Underground Mathematics | More challenging direct proof with worked solutions. | p1 |
| [Differentiation from](http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-firstppls-2009-1.pdf)[first principles](http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-firstppls-2009-1.pdf) | Math Centre | First principles introduction and examples. How to prove a derivative using first principles. | p1 |
| [First Principles worksheet](http://www.la-citadelle.com/courses/calculus/lim12.pdf) | La Citadelle, Canada | Student practice using first principles to find the derivative function (with answers). | p1 |
| [Mean and Variance of Binomial Random Variables](http://www.math.ubc.ca/~feldman/m302/binomial.pdf) | The University of British Columbia. | Proofs of E(x) = np and Var (x) = np(1-p) for the Binomial Distribution in statistics. Stretch activity. | p1 |
| [Introduction to Graphs](http://homepage.divms.uiowa.edu/~hzhang/c19/ch11.pdf) | University of Iowa | Informal approach to investigate ideas and attempt to establish results or proofs. Some ideas of disproof. Could be an enrichment task. Links to Graph Theory | p1 |
| [Handshaking and Chasing Kids](http://www.math.ucla.edu/~radko/circles/lib/data/Handout-433-532.pdf) | UCLA | More challenging work which is probably better suited to further maths students. | p1 |
| [A Geometric Proof of Heron's Formula](http://jwilson.coe.uga.edu/EMT668/EMAT6680.2000/Umberger/MATH7200/HeronFormulaProject/GeometricProof/geoproof.html) | University of Georgia | This resource is a more complex geometric proof given in full detail. This work provides greater challenge for students. | p1 |
| [Given two lines crossing a circle, can we show these lengths are equal?](https://undergroundmathematics.org/circles/r9551) | Underground Mathematics | Coordinate Geometry Proof, with hints and worked solution. | p1 |
| [Euler’s Formula & Platonic Solids](https://sites.math.washington.edu/~julia/teaching/445_Spring2013/Euler_Presentation.pdf)  | University of Washington | Introductory Graphy theory notes and leads into the proof that there are only 5 Platonic Solids. | p1 |
| [Proof: A Brief Historical Survey](https://nrich.maths.org/5996). | nrich | Brief history of mathematical proof. This may be useful as a document to help students review the topic of proof. It may also be a useful revision aid; students could read and highlight areas of useful guidance before they begin past paper practice. | p1, p3 |
| [Surfaces and Topology - Professor Raymond Flood](https://www.youtube.com/watch?v=pwT0Aaesp1M) | Gresham College | A lecture which lasts for 57 minutes which includes a well explained proof of Euler’s Polyhedron Formula V-E+F=2 for convex polyhedra. This proof uses a lot of linguistic reasoning in addition to algebra and numbers. The video also contains other proofs. | p1 |
| [A Tribute to Euler - William Dunham](https://www.youtube.com/watch?v=fEWj93XjON0) | Clay Maths Institute | 55 minute lecture held by the Clay Maths Institute in the US. Some good accessible content plus a proof concerning the partitioning of numbers.  | p1 |
| [Napoleon's Theorem](http://nrich.maths.org/1944) | nrich | Interactive game with a linked proof to demonstrate. Worked solution provided. | p1 |
| [Why I teach the identity symbol](https://www.tes.com/news/blog/why-i-teach-identity-symbol) | TES | Newspaper article from the TES which discusses the use of the identity symbol ≡ . Gives examples and explains how this symbol means that the statement is true for all values of a variable. Whereas the equals sign will only take particular values.  | p1 |
| [Logic and Mathematical Statements](http://www.math.toronto.edu/preparing-for-calculus/3_logic/we_2_if_then.html) | University of Toronto | This resource is introductory written notes and examples in addition to a video link from Toronto University. Good starter activity. | p1 |
| [Proof Strategies](https://math.dartmouth.edu/~m24s14/strategies.pdf) | Dartmouth College | Six page document on proof. Introduces some of the key ideas within logic and then gives numerous examples with full proofs. | p1, p2, and p3  |
| [Lines Intersecting Inside or Outside a Circle](http://jwilson.coe.uga.edu/emt668/EMAT6680.2003.fall/Nichols/6690/Webpage/Day%207.htm) | University of Georgia | Geometric Proof of when two lines intersect in a circle. | p1 |
| [Lengths of Segments of Chords](http://jwilson.coe.uga.edu/emt668/EMAT6680.2003.fall/Nichols/6690/Webpage/Day%208.htm) | University of Georgia | Geometric Proof of the relationship between the lengths of the four segments that are formed when two chords intersect in the interior of a circle. | p1 |
| [Segments of Tangents and Secants](http://jwilson.coe.uga.edu/emt668/EMAT6680.2003.fall/Nichols/6690/Webpage/Day%209.htm) | University of Georgia | Proof of relationship between two secant segments. | p1 |
| [Iff](http://nrich.maths.org/790) | nrich | Puzzle and worked solution of a proof of an if and only if statement. Could be used for flipped learning with more able students. | p1 |
| [Iffy Logic](https://nrich.maths.org/6331) | nrich | Interactive and accessible online or table game which practises the use of if … then and if and only if.Provides challenge. Ideal for interactive work in class. Ideal for group work. | p1 |
| [Answers to Proof Homework sheet](http://www.math.vt.edu/people/mcquain/sol_proofs_3.1-3.4.pdf) | **Virginia Polytechnic Institute and State University** | Worksheet with answers provided; includes some iff proofs. If used as a student exercise then the answers need to be separated from the questions. | p1, p2, p3 |
| [Proof Solutions: Inclass worksheet](http://www.math.vt.edu/people/mcquain/sol_proofs.pdf) | **Virginia Polytechnic Institute and State University** | Worksheet with answers provided; includes two iff proofs. If used as a student exercise then the answers need to be separated from the questions. | p1, p2, p3 |
| [If and only if](http://zimmer.csufresno.edu/~larryc/proofs/proofs.ifandonlyif.html) | **California State****University, Fresno** | Examples proving if and only if statements. More challenge. | p1 |
| [Disproof by Counterexample](https://cs.brown.edu/courses/cs022/static/files/documents/templates/counterexample.pdf) | Brown University | Single formal example of disproof by counter example. Extremely accessible. | p2 |
| [Legs Eleven](http://nrich.maths.org/564) | nrich | A good starter activity that is very accessible. | p1 and p2 |
| [On the Importance of Pedantry](http://nrich.maths.org/7665) | nrich | Some notes on Proof for Key Stage 4 and 5 teachers and learners | p2 |
| [To Prove or Not to Prove](http://nrich.maths.org/1387) | nrich | Some notes on Proof for Key Stage 4 and 5 teachers and learners. | p1  |
| [An Introduction to Proof by Contradiction](http://nrich.maths.org/4717) | nrich | Some notes on Proof for Key Stage 4 and 5 teachers and learners. | p3 |
| [Worksheet -disproof by counter example](http://www.nlcsmaths.com/uploads/2/6/3/6/26365454/o3qpro_a.pdf) | North London Collegiate School; Solomon Papers. | Lots of questions for students to grapple with in this worksheet. | p2 |
| [Garfield proof of Pythagoras](https://www.geogebra.org/m/PwUTN4bV) | Geogebra | Americans attribute this proof to President James Garfield.  | p1 |

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