ADVANCED GCE **4777/01**

**MATHEMATICS (MEI)**

Numerical Computation

**WEDNESDAY 18 JUNE 2008** Morning

Time: 2 hour 30 minutes

**Additional materials:** Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

*CUP/T39242*

**INSTRUCTIONS TO CANDIDATES**

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- Additional sheets, including computer print-outs, should be fastened securely to the Answer Booklet.

**COMPUTING RESOURCES**

- Candidates will require access to a computer with a spreadsheet program and suitable printing facilities throughout the examination.

**INFORMATION FOR CANDIDATES**

- The number of marks for each question is given in brackets [ ] at the end of each question or part question.
- In each of the questions you are required to write spreadsheet routines to carry out various numerical analysis processes. You should note the following points.
- You will not receive credit for using any numerical analysis functions which are provided within the spreadsheet. For example, many spreadsheets provide a solver routine; you will not receive credit for using this routine when asked to write your own procedure for solving an equation.

  You may use the following built-in mathematical functions: square root, sin, cos, tan, arcsin, arccos, arctan, ln, exp.
- For each question you attempt, you should submit print-outs showing the spreadsheet routine you have written and the output it generates. It will be necessary to print out the *formulae* in the cells as well as the *values* in the cells.

  You are not expected to print out and submit everything your routine produces, but you are required to submit sufficient evidence to convince the examiner that a correct procedure has been used.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

  

**1** **(i)** Explain carefully what it means to say that an iteration has first order convergence.
Show that, if $y_0$, $y_1$, $y_2$ are three terms in a first order iteration converging to $\alpha$, then $\alpha$ may be estimated as $\dfrac{y_1{}^2 - y_0 y_2}{2y_1 - y_0 - y_2}$. [6]

A curve has equation $x^y + y^x = 2$, where $x > 0$ and $y > 0$. Note that the point (1, 1) lies on the curve and that the curve is symmetrical about the line $y = x$. You are given that, for any value of $y$, there is only one value of $x$.

**(ii)** Show that, for $x = 1.1$, the equation may be re-arranged as $y = (2 - 1.1^y)^{\frac{1}{1.1}}$. Set up a spreadsheet to perform the iteration based on this rearrangement. Starting with $y_0 = 1$, obtain $y_1$ and $y_2$. Use the result in part **(i)** to obtain a more accurate value of $y$ when $x = 1.1$.

Repeat this process beginning with the more accurate value of $y$. Comment on the likely accuracy of your new estimate. [6]

**(iii)** Repeat the process in part **(ii)** to obtain estimates of $y$ for $x = 1.2, 1.3, \ldots 2.0$.
Comment on the likely accuracy of your result for $x = 2$.

Use the spreadsheet to obtain a sketch of the curve for $0 < x \leqslant 2$, $0 < y \leqslant 2$. [12]

**2** The trapezium rule, using $n$ strips of width $h$, is used to find an estimate $T_n$ of the integral

$$I = \int_a^b \mathrm{f}(x)\,\mathrm{d}x,$$

where $b - a = nh$. You may assume that the global error in $T_n$ is of the form

$$A_2 h^2 + A_4 h^4 + A_6 h^6 + \ldots,$$

where the coefficients $A_2, A_4, A_6, \ldots$ are independent of $n$ and $h$.

**(i)** Show that $T_n{}^* = \dfrac{4T_{2n} - T_n}{3}$ is an estimate of $I$ with global error of order $h^4$.

Write down an expression, $T_n{}^{**}$, in terms of $T_{2n}{}^*$ and $T_n{}^*$, that represents an estimate of $I$ with global error of order $h^6$. [6]

**(ii)** Use Romberg's method on a spreadsheet to find the value of

$$I = \int_0^2 \frac{x^2}{1 + \mathrm{e}^{-x}}\,\mathrm{d}x$$

correct to 6 decimal places. [9]

**(iii)** Modify your spreadsheet to find the value of

$$J = \int_0^k \frac{x^2}{1 + \mathrm{e}^{-x}}\,\mathrm{d}x$$

for $k = 0, 0.25, \ldots, 2$. Hence obtain a sketch of $J$ against $k$. [6]

**(iv)** Use your spreadsheet to determine, correct to 2 decimal places, the value of $k$ for which $J = 1$. [3]

**3** The differential equation

$$\frac{dy}{dx} = 1 - \sqrt{x+y}, \text{ with } y = 0 \text{ when } x = 0,$$

is to be solved numerically.

**(i)** Use the Runge-Kutta order 4 method with $h = 0.2$ to obtain a sketch of the solution curve for $0 < x < 3$. Give a rough estimate of the coordinates of the turning point $(p, q)$ on the solution curve. Also give a rough estimate of $\alpha$, the value of $x$ for which the curve crosses the horizontal axis. [11]

**(ii)** By reducing $h$ appropriately, obtain the values of $p$, $q$ and $\alpha$ correct to 2 decimal places. [5]

**(iii)** The differential equation is now generalised to

$$\frac{dy}{dx} = s - \sqrt{x+y}, \text{ with } y = 0 \text{ when } x = 0.$$

Modify your spreadsheet to find, correct to 2 decimal places, the value of $s$ for which $\alpha = 1$. [8]

**4** A curve of the form

$$y = a + bx + cx^2 \qquad (1)$$

is to be fitted, using least squares, to a set of data points $(x_i, y_i)$, $i = 1, 2, \ldots, n$.

**(i)** Show, using partial differentiation, that one of the normal equations is

$$\Sigma y = na + b\,\Sigma x + c\,\Sigma x^2.$$

Write down the other two normal equations. [5]

**(ii)** Use a spreadsheet to obtain a scatter diagram for the following data.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x_i$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
| $y_i$ | 1.02 | 2.08 | 2.73 | 3.14 | 2.87 | 2.22 | 1.43 |

What feature of the data suggests that a curve of the form (1) might be a suitable fit? [3]

**(iii)** Use a spreadsheet to

(*A*) formulate the normal equations,

(*B*) solve for $a$, $b$, $c$, using Gaussian elimination,

(*C*) find, and comment on, the sum of the residuals,

(*D*) find the residual sum of squares. [16]