OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4756

Further Methods for Advanced Mathematics (FP2)

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Friday 9 January 2009
Morning

Duration: 1 hour 30 minutes

4756

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions in Section A and **one** question from Section B.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Section A (54 marks)

Answer all the questions

1 **(a)** **(i)** By considering the derivatives of $\cos x$, show that the Maclaurin expansion of $\cos x$ begins

$$1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4.$$ **[4]**

(ii) The Maclaurin expansion of $\sec x$ begins

$$1 + ax^2 + bx^4,$$

where a and b are constants. Explain why, for sufficiently small x,

$$\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4\right)\left(1 + ax^2 + bx^4\right) \approx 1.$$

Hence find the values of a and b. **[5]**

(b) **(i)** Given that $y = \arctan\left(\dfrac{x}{a}\right)$, show that $\dfrac{dy}{dx} = \dfrac{a}{a^2 + x^2}$. **[4]**

(ii) Find the exact values of the following integrals.

(A) $\displaystyle\int_{-2}^{2} \frac{1}{4 + x^2}\, dx$ **[3]**

(B) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{4}{1 + 4x^2}\, dx$ **[3]**

2 **(i)** Write down the modulus and argument of the complex number $e^{j\pi/3}$. **[2]**

(ii) The triangle OAB in an Argand diagram is equilateral. O is the origin; A corresponds to the complex number $a = \sqrt{2}(1 + j)$; B corresponds to the complex number b.

Show A and the two possible positions for B in a sketch. Express a in the form $re^{j\theta}$. Find the two possibilities for b in the form $re^{j\theta}$. **[5]**

(iii) Given that $z_1 = \sqrt{2}e^{j\pi/3}$, show that $z_1^6 = 8$. Write down, in the form $re^{j\theta}$, the other five complex numbers z such that $z^6 = 8$. Sketch all six complex numbers in a new Argand diagram. **[6]**

Let $w = z_1 e^{-j\pi/12}$.

(iv) Find w in the form $x + jy$, and mark this complex number on your Argand diagram. **[3]**

(v) Find w^6, expressing your answer in as simple a form as possible. **[2]**

3 **(a)** A curve has polar equation $r = a\tan\theta$ for $0 \leqslant \theta \leqslant \frac{1}{3}\pi$, where a is a positive constant.

(i) Sketch the curve. **[3]**

(ii) Find the area of the region between the curve and the line $\theta = \frac{1}{4}\pi$. Indicate this region on your sketch. **[5]**

(b) **(i)** Find the eigenvalues and corresponding eigenvectors for the matrix $\mathbf{M}$ where

$$\mathbf{M} = \begin{pmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{pmatrix}.$$ **[6]**

(ii) Give a matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M} = \mathbf{QDQ}^{-1}$. **[3]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(a)** **(i)** Prove, from definitions involving exponentials, that

$$\cosh^2 x - \sinh^2 x = 1.$$ **[2]**

(ii) Given that $\sinh x = \tan y$, where $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$, show that

(A) $\tanh x = \sin y$,

(B) $x = \ln(\tan y + \sec y)$. **[6]**

(b) **(i)** Given that $y = \operatorname{artanh} x$, find $\dfrac{dy}{dx}$ in terms of x.

Hence show that $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1-x^2}\,dx = 2\operatorname{artanh}\tfrac{1}{2}$. **[4]**

(ii) Express $\dfrac{1}{1-x^2}$ in partial fractions and hence find an expression for $\displaystyle\int \frac{1}{1-x^2}\,dx$ in terms of logarithms. **[4]**

(iii) Use the results in parts **(i)** and **(ii)** to show that $\operatorname{artanh}\frac{1}{2} = \frac{1}{2}\ln 3$. **[2]**

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 The limaçon of Pascal has polar equation $r = 1 + 2a\cos\theta$, where a is a constant.

(i) Use your calculator to sketch the curve when $a = 1$. (You need not distinguish between parts of the curve where r is positive and negative.) **[3]**

(ii) By using your calculator to investigate the shape of the curve for different values of a, positive and negative,

(A) state the set of values of a for which the curve has a loop within a loop,

(B) state, with a reason, the shape of the curve when $a = 0$,

(C) state what happens to the shape of the curve as $a \to \pm\infty$,

(D) name the feature of the curve that is evident when $a = 0.5$, and find another value of a for which the curve has this feature. **[7]**

(iii) Given that $a > 0$ and that a is such that the curve has a loop within a loop, write down an equation for the values of θ at which $r = 0$. Hence show that the angle at which the curve crosses itself is $2\arccos\left(\frac{1}{2a}\right)$.

Obtain the cartesian equations of the tangents at the point where the curve crosses itself. Explain briefly how these equations relate to the answer to part **(ii)**(A). **[8]**