RECOGNISING ACHIEVEMENT

ADVANCED GCE **4756/01**

MATHEMATICS (MEI)

Further Methods for Advanced Mathematics (FP2)

THURSDAY 15 MAY 2008 Morning

Time: 1 hour 30 minutes

Additional materials (enclosed): None

Additional materials (required):
Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions in Section A and **one** question from Section B.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

[Turn over

Section A (54 marks)

Answer all the questions

1 **(a)** A curve has cartesian equation $(x^2 + y^2)^2 = 3xy^2$.

(i) Show that the polar equation of the curve is $r = 3\cos\theta\sin^2\theta$. [3]

(ii) Hence sketch the curve. [3]

(b) Find the exact value of $\displaystyle\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx$. [5]

(c) **(i)** Write down the series for $\ln(1+x)$ and the series for $\ln(1-x)$, both as far as the term in x^5. [2]

(ii) Hence find the first three non-zero terms in the series for $\ln\left(\dfrac{1+x}{1-x}\right)$. [2]

(iii) Use the series in part **(ii)** to show that $\displaystyle\sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r} = \ln 3$. [3]

2 You are given the complex numbers $z = \sqrt{32}(1+\mathrm{j})$ and $w = 8\left(\cos\frac{7}{12}\pi + \mathrm{j}\sin\frac{7}{12}\pi\right)$.

(i) Find the modulus and argument of each of the complex numbers z, z^*, zw and $\dfrac{z}{w}$. [7]

(ii) Express $\dfrac{z}{w}$ in the form $a + b\mathrm{j}$, giving the exact values of a and b. [2]

(iii) Find the cube roots of z, in the form $r\mathrm{e}^{\mathrm{j}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [4]

(iv) Show that the cube roots of z can be written as

$$k_1 w^*, \quad k_2 z^* \quad \text{and} \quad k_3 \mathrm{j}w,$$

where k_1, k_2 and k_3 are real numbers. State the values of k_1, k_2 and k_3. [5]

3 **(i)** Given the matrix $\mathbf{Q} = \begin{pmatrix} 2 & -1 & k \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ (where $k \neq 3$), find $\mathbf{Q}^{-1}$ in terms of k.

Show that, when $k = 4$, $\mathbf{Q}^{-1} = \begin{pmatrix} -1 & 6 & -1 \\ 1 & -8 & 2 \\ 1 & -5 & 1 \end{pmatrix}$. [6]

The matrix $\mathbf{M}$ has eigenvectors $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$, with corresponding eigenvalues 1, -1 and 3 respectively.

(ii) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{MP} = \mathbf{D}$, and hence find the matrix $\mathbf{M}$. [7]

(iii) Write down the characteristic equation for $\mathbf{M}$, and use the Cayley-Hamilton theorem to find integers a, b and c such that $\mathbf{M}^4 = a\mathbf{M}^2 + b\mathbf{M} + c\mathbf{I}$. [5]

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, prove that

$$\cosh^2 x - \sinh^2 x = 1.$$ [3]

(ii) Solve the equation $4\cosh^2 x + 9\sinh x = 13$, giving the answers in exact logarithmic form. [6]

(iii) Show that there is only one stationary point on the curve

$$y = 4\cosh^2 x + 9\sinh x,$$

and find the y-coordinate of the stationary point. [4]

(iv) Show that $\int_0^{\ln 2} (4\cosh^2 x + 9\sinh x)\,dx = 2\ln 2 + \frac{33}{8}$. [5]

[Question 5 is printed overleaf.]

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 A curve has parametric equations $x = \lambda \cos\theta - \dfrac{1}{\lambda}\sin\theta$, $y = \cos\theta + \sin\theta$, where λ is a positive constant.

(i) Use your calculator to obtain a sketch of the curve in each of the cases

$$\lambda = 0.5, \quad \lambda = 3 \quad \text{and} \quad \lambda = 5.$$ [3]

(ii) Given that the curve is a conic, name the type of conic. [1]

(iii) Show that y has a maximum value of $\sqrt{2}$ when $\theta = \frac{1}{4}\pi$. [2]

(iv) Show that $x^2 + y^2 = (1 + \lambda^2) + \left(\dfrac{1}{\lambda^2} - \lambda^2\right)\sin^2\theta$, and deduce that the distance from the origin of any point on the curve is between $\sqrt{1 + \dfrac{1}{\lambda^2}}$ and $\sqrt{1 + \lambda^2}$. [6]

(v) For the case $\lambda = 1$, show that the curve is a circle, and find its radius. [2]

(vi) For the case $\lambda = 2$, draw a sketch of the curve, and label the points A, B, C, D, E, F, G, H on the curve corresponding to $\theta = 0, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi, \pi, \frac{5}{4}\pi, \frac{3}{2}\pi, \frac{7}{4}\pi$ respectively. You should make clear what is special about each of these points. [4]