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Examiners' report

FURTHER MATHEMATICS A

H245

For first teaching in 2017

Y545/01 Summer 2023 series

Version 1

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Introduction

Our examiners' reports are produced to offer constructive feedback on candidates' performance in the examinations. They provide useful guidance for future candidates.

The reports will include a general commentary on candidates' performance, identify technical aspects examined in the questions and highlight good performance and where performance could be improved. A selection of candidate answers are also provided. The reports will also explain aspects which caused difficulty and why the difficulties arose, whether through a lack of knowledge, poor examination technique, or any other identifiable and explainable reason.

Where overall performance on a question/question part was considered good, with no particular areas to highlight, these questions have not been included in the report.

A full copy of the question paper and the mark scheme can be downloaded from OCR.

Paper Y545/01 series overview

Additional Pure Maths is one of the optional components and was sat by around one quarter of the cohort of Further Maths A – H245 in this summer series. A Level Further Maths was first assessed in 2019 and this is only the third summer exam series. Given that the majority of this cohort experienced disruptions to key stage 4 teaching and learning and most did not sit high stake external examinations over the past two consecutive years, the quality of the responses received was very promising indeed.

Although greater detail will follow below in the comments on individual questions, there are a number of broad areas to be mentioned for the benefit of future candidates. As in past series, questions requiring explanation, justification or supporting reasoning were generally handled variably. A significant number of candidates struggled to cover the key point(s) within a written response; many candidates often managed to say something correct, but were unable to state convincing mathematical reasons for their observation(s) and so could not be given (full) credit. This was especially so in Question 2 (b) (calculus), Question 3 (b) (ii) (vector product), Question 4 (sequences) and Question 7 (b) (iii) (Fibonacci and related numbers) as well as, to a lesser extent, in Question 5 (a) (ii) and Question 6 (b). Reinforcing the understanding of mathematical terms such as 'isomorphic', 'identity', 'inverse', and 'closure', and honing candidates' skills in conjecturing and proving will increase their confidence and improve a pleasing set of results even further.

Candidates who did well on this paper generally:	**Candidates who did less well on this paper generally:**
• were able to use and apply standard techniques with confidence • did well in Questions 7 (b) and 9 (a), (b) and (c), on topics which have been regularly assessed in the past and could then be practised via past papers • were able to apply their problem solving skills to translate problems in both mathematical and non-mathematical contexts into mathematical processes (see for instance Questions 7 and 2 (b), respectively).	• made arithmetic or algebraic errors which prevented them from gaining some of the more procedural marks on the paper (for example in Questions 2, 3 and 6) • struggled to correctly start and carry out proofs of all kinds (for example in Questions 5, 7 and 8) • struggled to deal with problem solving applied to sequences in Question 4 and to justify the nature of the saddle point in Question 6.

Question 1 (a) and (b)

1 The surface S is defined for all real x and y by the equation $z = x^2 + 2xy$. The intersection of S with the plane Π gives a section of the surface. On the axes provided in the Printed Answer Booklet, sketch this section when the equation of Π is each of the following.

(a) $x = 1$ **[2]**

(b) $y = 1$ **[2]**

The vast majority of candidates answered Question 1 well. A few candidates did not label axes and/or intercepts with axes. As general advice, we would recommend that the line or curve is also labelled with its equation, though the lack of the latter was not penalised. In this question, one strand of AO2 (make deductions) is tested.

Question 2 (a) and (b)

2 A curve has equation $y = \sqrt{1 + x^2}$, for $0 \leqslant x \leqslant 1$, where both the x- and y-units are in cm. The area of the surface generated when this curve is rotated fully about the x-axis is $A\,\text{cm}^2$.

(a) Show that $A = 2\pi \int_0^1 \sqrt{1 + kx^2}\,dx$ for some integer k to be determined. **[4]**

A small component for a car is produced in the shape of this surface. The curved surface area of the component must be $8\,\text{cm}^2$, accurate to within one percent. The engineering process produces such components with a curved surface area accurate to within one half of one percent.

(b) Determine whether all components produced will be suitable for use in the car. **[2]**

Most candidates gained full marks in part (a), demonstrating excellent knowledge of area of surface of revolution and command of the chain rule. In this question part only AO1 (use and apply standard techniques) is assessed.

In part (b), less than half of the candidates obtained any mark. Most of the candidates evaluated the required integral by calculator, as expected. In this question, where AO3 is assessed (translate problems in non-mathematical contexts into mathematical processes), a clear numerical comparison was required, with the bounds for both the available material and the engineering model explicitly given.

Assessment for learning

'Determine' indicates that justification should be given for any results found, which includes use of the correct notation of integrals, such as dx and limits. It does not prohibit the use of calculators to offload computation but does mean that it must be clear where any numbers have come from, what they represent and, as in part (b) how they are to be used in finding the solution.

Question 3 (a), (b) (i) and (b) (ii)

3 The points A and B have position vectors $\mathbf{a} = \mathbf{i} + p\mathbf{j} + q\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ respectively, relative to the origin O.

(a) Determine the value of p and the value of q for which $\mathbf{a} \times \mathbf{b} = 2\mathbf{i} + 6\mathbf{j} - 11\mathbf{k}$. **[3]**

(b) The point C has coordinates (d, e, f) and the tetrahedron $OABC$ has volume 7.

(i) Using the values of p and q found in part **(a)**, find the possible relationships between d, e and f. **[2]**

(ii) Explain the geometrical significance of these relationships. **[2]**

This question focused on theoretical vectors.

In part (b) (i), the vast majority candidates knew the formula for the volume of a tetrahedron in terms of the scalar triple product. A small minority of candidates did not provide two relationships between d, e and f.

In part (b) (ii), a majority of candidates obtained no marks, showing that further work is required on understanding how planes in their general algebraic forms are related to the geometrical forms in 3D space. A minority of candidates recognised that the equations found in part (b) (i) describe two planes, thereby gaining 1 mark, but very few were able to articulate that such planes are parallel to that of *OAB.*

Part (b) of this question assessed two of AO2 strands, make deductions and use mathematical language and notation correctly.

Misconception

A small number of candidates calculated a different scalar triple product, other than $(a \times b) \cdot c$, to obtain the equation of the second plane, rather than using $\frac{1}{6}|(a \times b) \cdot c| = 7$.

Question 4

4 The sequence $\{A_n\}$ is given for all integers $n \geqslant 0$ by $A_n = \frac{I_{n+2}}{I_n}$, where $I_n = \int_0^{\frac{1}{2}\pi} \cos^n x \, dx$.

- Show that $\{A_n\}$ increases monotonically.
- Show that $\{A_n\}$ converges to a limit, A, whose exact value should be stated. **[7]**

This question assessed two of AO2 strands, construct rigorous mathematical arguments (including proofs) and make deductions and therefore it was essential that candidates presented their reasoning in a rigorous manner to obtain full marks. It also contains 2 marks of AO3, translate problems in mathematical contexts into mathematical processes and interpret solutions to problems in their original context, which introduce an element of challenge for many candidates.

This question proved to be a good discriminator, with candidates scoring fully, gaining 4 out of the 7 marks, gaining 2 or 3 marks, or not gaining any marks in roughly equal proportions.

In general, the use of reduction formulae and integration by parts was good, though some candidates needed more fluency in applying the chain rule. About half of the candidates showed evidence of some knowledge of monotonically increasing sequences, but only a minority of the candidates gaining between 5 and 7 marks were able to use algebraic techniques to actually show that A_n is monotonic increasing. Likewise, about half of the candidates understood that $A = 1$, but a minority used an algebraic technique to show how they had arrived at the value of the limit and had formal, fully correct notation.

Assessment for learning

Candidates should be exposed to discussions on more complex limits and encouraged to use the correct notation, avoiding for example expressions like $\frac{infinity}{infinity}$ or $\frac{n+1}{n+2} = \frac{L}{L} = 1$ as $n \to \infty$.

Exemplar 1

$I_n = \int_0^{\frac{\pi}{2}} \cos^n x = \cos^{n-1} x \times \cos x$

$u = \cos^{n-1} x \qquad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = (n-1) \times -\sin x \times \cos^{n-2} x \quad V = \sin x$

$\left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \sin^2 x \cos^{n-2} x$

$+ (n-1) \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \cos^{n-2} x$

$+ (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x - \cos^n x$

$I_n = 0 + (n-1) I_{n-2} - (n-1) I_n$

$n I_n = (n-1) I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}$

$I_{n+2} = \frac{n+2-1}{n+2} I_n$

$I_{n+2} = \frac{n+1}{n+2} I_n$

$\frac{2}{3}, \frac{3}{4}$

$\frac{\frac{n+1}{n+2} I_n}{I_n} = A_n = \frac{n+1}{n+2}$

$\{A_n\}$ converges to 1 as n gets large

$n \geq 0$

$\frac{n+2}{n+3} - \frac{n+1}{n+2} = \frac{(n+2)^2 - (n+1)(n+3)}{(n+2)(n+3)} = \frac{1}{(n+2)(n+3)}$

difference is +ve $= \frac{1}{(n+2)(n+3)}$ so monotonic increasing as n+1 term > n term

In this response the candidate gains 6 of the 7 marks available; one mark is lost as the value of the limit is not arrived at using an algebraic technique. Four marks are gained for deriving the reduction formula using integration by parts; further 2 marks are gained for convincingly showing that A_n is monotonic increasing.

Question 5 (a) (i) and (ii)

5 **(a)** The group G consists of the set $S = \{1, 9, 17, 25\}$ under $\times_{32}$, the operation of multiplication modulo 32.

(i) Complete the Cayley table for G given in the Printed Answer Booklet. **[2]**

(ii) Up to isomorphisms, there are only two groups of order 4.

- C_4, the cyclic group of order 4
- K_4, the non-cyclic (Klein) group of order 4

State, with justification, to which of these two groups G is isomorphic. **[2]**

Question 5 (a) (i) was generally well answered.

In Question 5 (a) (ii), the vast majority of candidates correctly identified C_4 as the isomorphic group. Most of these adequately justified their selection by referring to the two elements of order four as generators, while those that didn't merely stated the order of the elements. Very few used negative justification, explaining why the isomorphic group could not be K_4. This part question assesses two strands of AO2, make deductions and explain reasoning.

Question 5 (b) (i) and (ii)

(b) (i) List the odd quadratic residues modulo 32. **[2]**

(ii) Given that n is an odd integer, prove that $n^6 + 3n^4 + 7n^2 \equiv 11 \pmod{32}$. **[4]**

In part (b) (i), nearly all candidates earned both marks, predominantly by evaluating the squares of (odd) numbers from 1 to 31 mod 32.

In part (b) (ii), candidates mainly either knew how to proceed and achieved full marks or were unsure of how to start and gained no marks at all. For the minority of candidates who obtained between 1 and 3 marks (out of 4), a common issue was not showing sufficiently that the given expression is 11 (mod 32) by exhaustion for each of the four values of n^2 (mod 32). In this question, AO3 is assessed (translate problems in mathematical contexts into mathematical processes), together with AO2 (make deductions and explain reasoning).

Assessment for learning

Candidates would benefit from solving problems in modular arithmetic in a variety of contexts.

OCR support

Useful resources are contained in the Delivery guide 8.02 Number Theory.

Misconception

A significant number of candidates substituted $n = 1, 9, 17, 25$ in their expression instead of n^2. Others calculated the correct values of n^4 and n^6 (mod 32) for $n^2 = 1, 9, 17, 25$, but substituted them in the given expression in the wrong order.

Question 6 (a), (b) and (c)

6 The surface S has equation $z = x\sin y + \frac{y}{x}$ for $x > 0$ and $0 < y < \pi$.

(a) Determine, as a function of x and y, the determinant of **H**, the Hessian matrix of S. **[6]**

(b) Given that S has just one stationary point, P, use the answer to part **(a)** to deduce the nature of P. **[2]**

(c) The coordinates of P are (α, β, γ).

Show that β satisfies the equation $\beta + \tan\beta = 0$. **[3]**

Question 6 was a routine question on this topic for most candidates, although a few did drop marks on part (b). The majority of candidates correctly identified the nature of the stationary point in part (b). A number of the candidates lost the second mark because, having expanded the square brackets in the determinant of the Hessian matrix in part (a), they were unable to solve the inequality to justify the nature of the stationary point. Naturally, this question contains 2 marks of AO2 (make deductions and explain reasoning).

Question 7 (a)

7 Binet's formula for the nth Fibonacci number is given by $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ for $n \geqslant 0$, where α and β (with $\alpha > 0 > \beta$) are the roots of $x^2 - x - 1 = 0$.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$. **[1]**

Candidates had no issues with part (a)

Question 7 (b) (i), (ii) and (iii)

(b) Consider the sequence $\{S_n\}$, where $S_n = \alpha^n + \beta^n$ for $n \geqslant 0$.

(i) Determine the values of S_2 and S_3. **[3]**

(ii) Show that $S_{n+2} = S_{n+1} + S_n$ for $n \geqslant 0$. **[2]**

(iii) Deduce that S_n is an integer for all $n \geqslant 0$. **[1]**

In part (b) (i), a significant number of candidates did not use their result of part (a) in their justification, opting instead to evaluate α and β from the given quadratic equation and then S_2 and S_3 by calculator, thereby risking failing to provide all the detail required by the command word 'Determine'. Nonetheless, most candidates managed to obtain full marks. The 2 marks available in this part question are an AO3 (translate problems in mathematical contexts into mathematical processes) and an AO2 (make deductions).

However, the strategy of using the exact values of α and β was often unsuccessful in part (b) (ii), where over half of candidates gained no marks of the 2 available. Indeed, a statement that $\alpha^2 = \alpha + 1$ (and an analogue one for β), seen or implied, was necessary to gain the first mark with this approach. Candidates who used the results of part (a) to express S_{n+2} in terms of S_n and S_{n+1} were almost guaranteed to achieve full marks.

In part (b) (iii), approximately half of candidates gained the single mark available. Given that in this part question AO2 is assessed (explain reasoning), a clear explanation of the inductive nature of the relation in part (b) (ii) to demonstrate that the sequence S_n generates only integers was required.

This question was meant to stretch and challenge, as reflected by its position in the question paper.

Question 7 (c) (i) and (ii)

(c) A student models the terms of the sequence $\{S_n\}$ using the formula $T_n = \alpha^n$.

(i) Explain why this formula is unsuitable for every $n \geqslant 1$. **[1]**

(ii) Considering the cases n even and n odd separately, state a modification of the formula $T_n = \alpha^n$, other than $T_n = \alpha^n + \beta^n$, such that $T_n = S_n$ for all $n \geqslant 1$. **[2]**

Most candidates were able to appreciate that the sequence T_n fails to give integers (AO3: recognise the limitations of models). However, a small number of candidates considered the INT function to provide an alternative formula for S_n (AO3: explain how to refine models).

OCR support

Candidates would benefit from further exposure to the use of the INT(x) function for discrete models. See for example:

- Question 7 in Sample paper
- Question 5 in Practice Paper - Set 1
- Question 6 in Question paper - June 2022

Question 8 (a) (i) and (ii)

8 Let $f(n)$ denote the base-n number 2121_n where $n \geqslant 3$.

(a) **(i)** For each $n \geqslant 3$, show that $f(n)$ can be written as the product of two positive integers greater than 1, $a(n)$ and $b(n)$, each of which is a function of n. **[2]**

(ii) Deduce that $f(n)$ is always composite. **[1]**

Most candidates performed well in part (a) (i), demonstrating their knowledge and ability to work with numbers written in base n, where n is a positive integer. The small number of candidates who obtained 1 mark of the 2 available mainly did not factorise their cubic expression correctly, using synthetic division and ending up with a non-integer factor $n + \frac{1}{2}$.

In part (a) (ii), almost half of candidates understood the requirement for compositeness of an integer and provided a clear explanation (AO2: explain reasoning), while the other half did not communicate the crucial information that the factors are not equal to 1 (or are greater than 1) or focused (only) on showing that the two factors are always different for $n \geq 3$.

Misconception

The notion that a positive integer is composite if it has at least one divisor other than one and itself should be recalled and incorporated in problem solving. Many candidates believed they had to show that the two factors $a(n)$ and $b(n)$ are different for all values of $n \geq 3$ to show that $f(n)$ is always composite.

Question 8 (b) (i)

(b) Let h be the highest common factor of $\mathrm{a}(n)$ and $\mathrm{b}(n)$.

(i) Prove that h is either 1 or 5. **[4]**

This stretch and challenge question proved challenging for most candidates, with many gaining no marks on this part. Slightly less than half gained at least 1 mark by attempting a linear combination of their two factors (AO3: translate problems in mathematical contexts into mathematical processes). In the remaining part of the question, three of AO2 strands are assessed: construct rigorous mathematical arguments (including proofs), make deductions and explain reasoning. A minority of candidates managed to reduce the problem to finding the highest common factor of two linear factors in n, thereby gaining the second mark, and only a small number managed to complete the proof. The alternative solution in the mark scheme, based on finding a single linear combination of the original factors which is equal to 5, appeared relatively frequently among complete solutions.

Exemplar 2

Let $h = \text{hcf}(2n+1, n^2+1)$

Then h divides:

$4(n^2+1) - (2n+1)(2n-1)$

$= 4n^2 + 4 - (4n^2 - 1)$

$= 5$

$\Rightarrow h \mid 5$

Hence $h = 1$ or 5.

The main purpose of this exemplar is to indicate that it is not really necessary to write lengthy responses. This succinct response contains all the key elements to gain each of the 4 marks available: a linear combination of the two factors is written and correctly evaluated, the highest common factor is stated to be a factor of 5 and finally established to be 1 or 5.

Question 8 (b) (ii)

(ii) Find a value of n for which $h = 5$. **[2]**

More than half of candidates did not gain any mark in this question as they gave an answer of $n = 2$, ignoring the condition $n \geq 3$ in the question. In this question AO3 (translate problems in mathematical contexts into mathematical processes) is assessed.

Question 9 (a)

9 The set C consists of the set of all complex numbers excluding 1 and -1. The operation $\oplus$ is defined on the elements of C by $a \oplus b = \dfrac{a+b}{ab+1}$ where $a, b \in C$.

(a) Determine the identity element of C under $\oplus$. **[2]**

The majority of candidates found the identity element correctly. A number of candidates merely verified that 0 is the identity, without deriving it formally, thereby losing one mark. In this question, AO2 is assessed (make deductions).

Assessment for learning

The command word 'Determine' signposts that the identity element has to be found by solving an equation, rather than verifying that an element *e* satisfies it.

Question 9 (b)

(b) For each element x in C show that it has an inverse element in C. **[2]**

Slightly more than half of the candidates were able to provide the correct inverse. The AO2 assessed in this question (construct rigorous mathematical arguments, including proofs) required a formal derivation to 'Show that'.

Question 9 (c)

(c) Show that $\oplus$ is associative on C. **[3]**

Most candidates gained at least 2 marks in this question, showing that the concept of associativity is generally well understood and proof of associativity well practised. A small number, having obtained the first 2 marks, did not correctly expand or simplify their expressions for comparison and conclusion.

Question 9 (d)

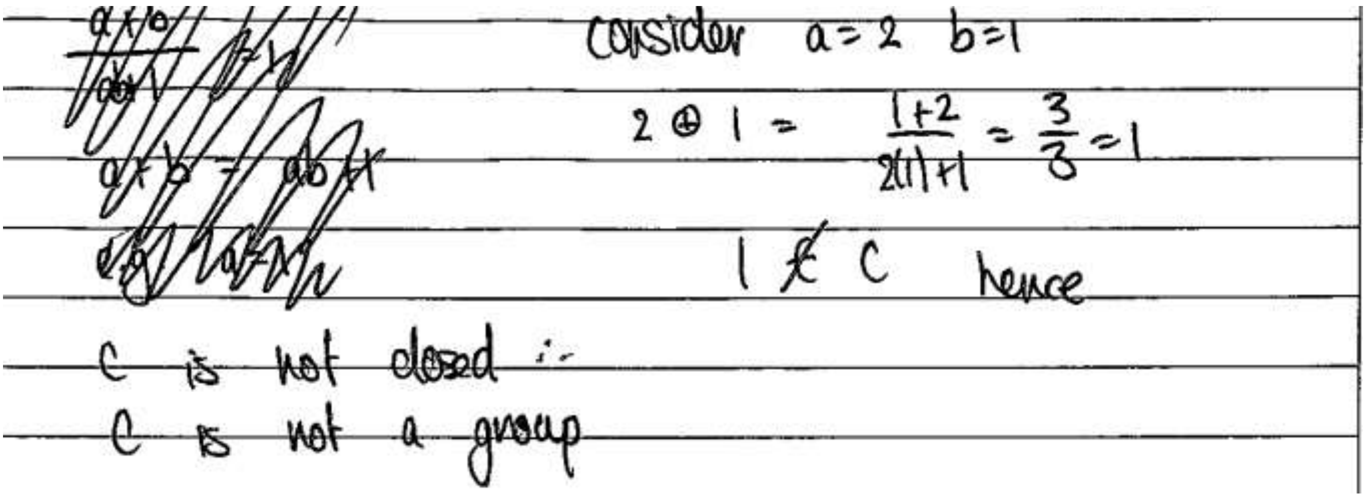
(d) Explain why $(C, \oplus)$ is not a group. **[1]**

A minority of candidates gained the single mark available in this question (AO2: assess the validity of mathematical arguments). Most candidates understood that closure does not apply, but did not provide a correct example. Many stated that $a \oplus b$ can be equal to ± 1, which do not belong to C, others that $a \oplus b$ is undefined when either a or b is 1 (which is not in C), still others that $a \oplus b$ can be real and thus not in C. A few candidates, having provided a correct example, stated that the set C is undefined, rather than closed.

Misconception

A number of candidates wrote that a real number does not belong to the set of complex numbers excluding 1 and -1.

Exemplar 3



This exemplar highlights a misconception which frequently appeared in responses. This candidate uses $b = 1$ to show lack of closure, which is not an element of C.

Question 9 (e)

(e) Find a subset, D, of C such that $(D, \oplus)$ is a group of order 3. **[3]**

Not all candidates made an attempt on this final part, but there was no evidence to suggest that those candidates making good progress on the earlier parts of this question specifically ran out of time. A minority of candidates achieved full marks, while the remaining candidates looked for closure of D and started using either $a \oplus a = -a$ or $a \oplus a \oplus a = 0$, but were either unable to solve for a or had errors in their working.

In this 'stretch and challenge' question, one strand of AO2 is assessed (construct rigorous mathematical arguments, including proofs), together with two strands of AO3 (translate problems in mathematical contexts into mathematical processes and interpret solutions to problems in their original context).

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